

Extra gauge symmetries in BHT gravity

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Abstract

We study the canonical structure of the Bergshoeff-Hohm-Townsend massive gravity, linearized around a maximally symmetric background. At the critical point in the space of parameters, defined by $\Lambda_0/m^2 = -1$, we discover an extra gauge symmetry, which reflects the existence of the partially massless mode. The number of the Lagrangian degrees of freedom is found to be 1. We show that the canonical structure of the theory at the critical point is unstable under linearization.

1 Introduction

Recently, Bergshoeff, Hohm and Townsend (BHT) proposed a parity conserving theory of gravity in three dimensions (3D), which is defined by adding certain curvature-squared terms to the Einstein-Hilbert action [1, 2]. When the BHT gravity is linearized around the Minkowski ground state, it is found to be equivalent to the Fierz-Pauli theory for a free massive spin-2 field [3]. Moreover, it is ghosts-free, unitary and renormalizable [4, 5]. On the other hand, the overall picture is changed when we go over to the (A)dS background, where various dynamical properties, such as unitarity, gauge invariance or boundary behavior, become more complex [2, 6, 7, 8, 9].

Dynamical characteristics of a gravitational theory take a particularly clear form in the constrained Hamiltonian approach [10]. Analyzing the nature of constraints in the fully *nonlinear* BHT gravity, we discovered the special role of an extra condition [11]; when applied to a maximally symmetric solution, it takes the familiar form $\Lambda_0/m^2 \neq -1$, where m^2 is the mass parameter and Λ_0 a cosmological constant¹. The resulting theory is found to possess two Lagrangian degrees of freedom, in agreement with the number of massive graviton modes on the (A)dS background [2]. In the present paper, we extend our investigation to the *critical point* $\Lambda_0/m^2 = -1$ in the maximally symmetric sector of the theory; in this case, the ground state is uniquely determined by an effective cosmological constant [2, 6]. In the linear approximation, there appears an *extra gauge invariance* which eliminates one component of the massive graviton, reducing it to the partially massless mode [13, 14, 15]. By comparing these results with those obtained nonperturbatively [11], we can understand

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¹The canonical analysis of the BHT gravity performed in [12] refers to the case $\Lambda_0 = 0$.

how the canonical structure of the BHT gravity is changed in the process of linearization. In Ref. [16], the canonical analysis of the linearized BHT gravity is carried out only for the generic values of the parameters.

The paper is organized as follows. In section 2, we give an account of the linearized BHT gravity in the Lagrangian formalism. In particular, we discuss the Lagrangian form of the extra gauge symmetry, constructed later by the canonical methods. In section 3, we perform a complete canonical analysis of the linearized BHT gravity around a maximally symmetric background, assuming the critical condition $\Lambda_0/m^2 = -1$. Then, in section 4, we classify the constraints and find a difference in their number and type (first or second class), in comparison to the results of the nonperturbative analysis [11]. As a consequence, we conclude that the theory exhibits a single Lagrangian degree of freedom. In section 5, the resulting set of constraints is used to construct the canonical generator of extra gauge symmetry. After that, the existing Lagrangian mode can be interpreted as a partially massless state. Finally, section 6 is devoted to concluding remarks, while appendices contain some technical details.

We use the same conventions as in Ref. [11]: the Latin indices refer to the local Lorentz frame, the Greek indices refer to the coordinate frame; the middle alphabet letters ($i, j, k, \dots; \mu, \nu, \lambda, \dots$) run over 0,1,2, the first letters of the Greek alphabet ($\alpha, \beta, \gamma, \dots$) run over 1,2; the metric components in the local Lorentz frame are $\eta_{ij} = (+, -, -)$; totally antisymmetric tensor ε^{ijk} and the tensor density $\varepsilon^{\mu\nu\rho}$ are both normalized by $\varepsilon^{012} = 1$.

2 Linearized Lagrangian dynamics

Following the approach defined in our previous paper [11], we study the BHT gravity in the framework of Poincaré gauge theory [17], where the basic gravitational variables are the triad field b^i and the Lorentz connection ω^k (1-forms), and the corresponding field strengths are the torsion $T^i = db^i + \varepsilon^i_{jk}\omega^j \wedge b^k$ and the curvature $R^i = d\omega^i + \frac{1}{2}\varepsilon^i_{jk}\omega^j \wedge \omega^k$ (2-forms). The underlying geometric structure corresponds to Riemann-Cartan geometry, in which b^i is an orthonormal coframe, $g := \eta_{ij}b^i \otimes b^j$ is the metric of spacetime, and ω^i is the Cartan connection. For $T_i = 0$, the geometry becomes Riemannian.

Lagrangian. In local coordinates x^μ , the BHT Lagrangian density can be written in the form [11]:

$$\mathcal{L} = a\varepsilon^{\mu\nu\rho} \left(\sigma b^i{}_\mu R_{i\nu\rho} - \frac{1}{3}\Lambda_0 \varepsilon^{ijk} b_{i\mu} b_{j\nu} b_{k\rho} \right) + \frac{a}{m^2} \mathcal{L}_K + \varepsilon^{\mu\nu\rho} \frac{1}{2} \lambda^i{}_\mu T_{i\nu\rho}. \quad (2.1a)$$

Here, the Lagrange multiplier $\lambda^i{}_\mu$ ensures the vanishing of torsion and thereby, the Riemannian nature of the connection, while \mathcal{L}_K is defined in terms of an auxiliary field $f^i{}_\mu$ as

$$\mathcal{L}_K = \frac{1}{2} \varepsilon^{\mu\nu\rho} f^i{}_\mu R_{i\nu\rho} - b \mathcal{V}_K, \quad \mathcal{V}_K = \frac{1}{4} (f_{i\mu} f^{i\mu} - f^2), \quad (2.1b)$$

where $f := f^k{}_\rho h_k{}^\rho$ and $b = \det(b^i{}_\mu)$. Using the field equations to eliminate $f^i{}_\mu$, one can verify that \mathcal{L}_K reduces to the standard BHT form.

Introducing the notation $Q_A = (b^i{}_\mu, \omega^i{}_\mu, f^i{}_\mu \lambda^i{}_\mu)$, we now consider the linearized form of the theory around a maximally symmetric solution \bar{Q}_A , characterized by (Appendix A)

$$\bar{G}_{ij} = \Lambda_{\text{eff}} \eta_{ij}, \quad \bar{f}^i{}_\mu = -\Lambda_{\text{eff}} \bar{b}^i{}_\mu, \quad \bar{\lambda}^i{}_\mu = 0, \quad (2.2a)$$

where Λ_{eff} is the effective cosmological constant. The linearization of the Lagrangian density (2.1) is based on the expansion

$$Q_A = \bar{Q}_A + \tilde{Q}_A, \quad (2.2b)$$

where \tilde{Q}_A is a small excitation around \bar{Q}_A . The piece of \mathcal{L} quadratic in \tilde{Q}_A takes the form:

$$\begin{aligned} \mathcal{L}^{(2)} = & a\varepsilon^{\mu\nu\rho} \left(2\sigma \tilde{b}^i{}_\mu \bar{\nabla}_\nu \tilde{\omega}_{i\rho} + \sigma \varepsilon^{ijk} \bar{b}^i{}_\mu \tilde{\omega}^j{}_\nu \tilde{\omega}^k{}_\rho - \Lambda_0 \varepsilon_{ijk} \bar{b}^i{}_\mu \tilde{b}^j{}_\nu \tilde{b}^k{}_\rho \right) \\ & + \frac{a}{m^2} \mathcal{L}_K^{(2)} + \varepsilon^{\mu\nu\rho} \tilde{\lambda}^i{}_\mu \left(\bar{\nabla}_\nu \tilde{b}_{i\rho} + \varepsilon_{ijk} \tilde{\omega}^j{}_\nu \bar{b}^k{}_\rho \right), \end{aligned} \quad (2.3a)$$

where

$$\begin{aligned} \mathcal{L}_K^{(2)} &:= \varepsilon^{\mu\nu\rho} \left(\tilde{f}^i{}_\mu \bar{\nabla}_\nu \tilde{\omega}_{i\rho} - \frac{\Lambda_{\text{eff}}}{2} \varepsilon_{ijk} \bar{b}^i{}_\mu \tilde{\omega}^j{}_\nu \tilde{\omega}^k{}_\rho \right) - (b\mathcal{V}_K)^{(2)}, \\ (b\mathcal{V}_K)^{(2)} &:= \frac{\bar{b}}{4} (\eta^{ij} \bar{g}^{\mu\nu} - \bar{h}^{i\mu} \bar{h}^{j\nu}) \tilde{f}_{i\mu} \tilde{f}_{j\nu} + \frac{\bar{b}}{2} \Lambda_{\text{eff}} (\eta^{ij} \bar{g}^{\mu\nu} + \bar{h}^{i\mu} \bar{h}^{j\nu} - 2\bar{h}^{i\nu} \bar{h}^{j\mu}) \tilde{f}_{i\mu} \tilde{b}_{j\nu} \\ &\quad + \frac{\bar{b}}{4} \Lambda_{\text{eff}}^2 (\eta^{ij} \bar{g}^{\mu\nu} - \bar{h}^{i\nu} \bar{h}^{j\mu}) \tilde{b}_{i\mu} \tilde{b}_{j\nu}. \end{aligned} \quad (2.3b)$$

Field equations. The variation of $\mathcal{L}^{(2)}$ with respect to $\tilde{Q}_A = (\tilde{b}^i{}_\mu, \tilde{\omega}^i{}_\mu, \tilde{f}^i{}_\mu, \tilde{\lambda}^i{}_\mu)$ yields the linearized BHT field equations:

$$\begin{aligned} a\varepsilon^{\mu\nu\rho} \left(2\sigma \bar{\nabla}_\nu \tilde{\omega}_{i\rho} - 2\Lambda_0 \varepsilon_{ijk} \bar{b}^j{}_\nu \tilde{b}^k{}_\rho \right) - \frac{a}{m^2} W_i{}^\mu + \varepsilon^{\mu\nu\rho} \bar{\nabla}_\nu \tilde{\lambda}_{i\rho} &= 0, \\ \varepsilon^{\mu\nu\rho} \left[a \bar{\nabla}_\nu \left(2\sigma \tilde{b}_{i\rho} + \frac{1}{m^2} \tilde{f}_{i\rho} \right) + a \left(2\sigma - \frac{\Lambda_{\text{eff}}}{m^2} \right) \varepsilon_{ijk} \bar{b}^j{}_\nu \tilde{\omega}^k{}_\rho + \varepsilon_{ijk} \bar{b}^j{}_\nu \tilde{\lambda}^k{}_\rho \right] &= 0, \\ \varepsilon^{\mu\nu\rho} \bar{\nabla}_\nu \tilde{\omega}_{i\rho} - \frac{\bar{b}}{2} \left[(\eta_{ij} \bar{g}^{\mu\nu} - \bar{h}_i{}^\mu \bar{h}_j{}^\nu) (\tilde{f}^j{}_\nu + \Lambda_{\text{eff}} \tilde{b}^j{}_\nu) + 2\Lambda_{\text{eff}} (\bar{h}_i{}^\mu \bar{h}_j{}^\nu - \bar{h}_i{}^\nu \bar{h}_j{}^\mu) \tilde{b}^j{}_\nu \right] &= 0, \\ \varepsilon^{\mu\nu\rho} \left(\bar{\nabla}_\nu \tilde{b}_{i\rho} + \varepsilon_{ijk} \tilde{\omega}^j{}_\nu \bar{b}^k{}_\rho \right) &= 0. \end{aligned} \quad (2.4)$$

where $W_i{}^\mu := \delta(b\mathcal{V}_K)^{(2)} / \delta \tilde{b}^i{}_\mu$ takes the form:

$$W_i{}^\mu = \frac{1}{2} \Lambda_{\text{eff}} \bar{b} \left[(\eta_{ij} \bar{g}^{\mu\nu} + \bar{h}_i{}^\mu \bar{h}_j{}^\nu - 2\bar{h}_i{}^\nu \bar{h}_j{}^\mu) \tilde{f}^j{}_\nu + \Lambda_{\text{eff}} (\eta_{ij} \bar{g}^{\mu\nu} - \bar{h}_i{}^\nu \bar{h}_j{}^\mu) \tilde{b}^j{}_\nu \right].$$

Let us now focus our attention on the trace of the first field equation, the linearized version of (A.3):

$$\left(\sigma + \frac{\Lambda_{\text{eff}}}{2m^2} \right) \bar{h}_i{}^\mu \left(\tilde{f}^i{}_\mu + \Lambda_{\text{eff}} \tilde{b}^i{}_\mu \right) = 0. \quad (2.5)$$

In the canonical approach, this relation is expected to be a *constraint*, as is the case in the nonlinear regime. However, there is a *critical condition* on parameters, defined by $\Lambda_{\text{eff}} + 2\sigma m^2 = 0$, for which equation (2.5) is identically satisfied. This is an important signal that the related canonical structure of the linearized theory might be significantly changed. Using (A.5), the critical condition can be equivalently written as

$$\Lambda_0/m^2 = -1, \quad (2.6)$$

or as $\Lambda_{\text{eff}} = 2\sigma\Lambda_0$. The central idea of our work is to examine the influence of this condition on the *canonical structure* of the linearized BHT massive gravity.

Extra gauge symmetry. When we have a maximally symmetric background, the critical condition (2.6) implies that the massive graviton of the linearized BHT gravity (with two helicity states) becomes a (single) partially massless mode; simultaneously, there appears an extra gauge symmetry in the theory. By a systematic analysis of the related canonical structure (see section 5), we discover that this symmetry has the following form:

$$\begin{aligned}\delta_E \tilde{b}^i{}_\mu &= \epsilon \bar{b}^i{}_\mu, \\ \delta_E \tilde{\omega}^i{}_\mu &= -\varepsilon^{ijk} \bar{b}_{j\mu} \bar{h}_k{}^\nu \bar{\nabla}_\nu \epsilon, \\ \delta_E \tilde{f}^i{}_\mu &= -2\bar{\nabla}_\mu (\bar{h}^{i\nu} \bar{\nabla}_\nu \epsilon) + \Lambda_{\text{eff}} \epsilon \bar{b}^i{}_\mu, \\ \delta_E \tilde{\lambda}^i{}_\mu &= 0,\end{aligned}\tag{2.7}$$

where ϵ is an infinitesimal gauge parameter. The proof of this statement at the level of the field equations (2.4) is given in Appendix B. Although the form of $\delta_E \tilde{f}^i{}_\mu$ has been known for some time, see for instance [15, 2], our result uncovers the very root of this symmetry by specifying its action on all the fields, including $\tilde{b}^i{}_\mu$. Up to second order terms, one can rewrite the infinitesimal gauge transformation of $\tilde{b}^i{}_\mu$ in the form $\delta_E \tilde{b}^i{}_\mu = \epsilon \bar{b}^i{}_\mu$, which looks like a Weyl rescaling. However, in doing so, one should keep in mind that (2.7) is *not* the symmetry of the full nonlinear theory, but only of its linearized version. Note also that the Weyl-like form of (2.7) closely resembles the result found in [18], which describes an extra gauge symmetry of the Chern-Simons gravity. The presence of the gauge parameter ϵ and its first and second derivatives in (2.7) indicates significant changes of the set of first class constraints, in comparison to the nonlinear BHT theory.

3 Canonical analysis of the linearized theory

We are now going to analyze the canonical structure of the BHT gravity linearized around the maximally symmetric background $G_{ij} = \Lambda_{\text{eff}} \eta_{ij}$, at the critical point (2.6). Technically, the analysis is based on the Lagrangian (2.3), quadratic in the excitation modes \tilde{Q}_A .

Primary constraints. If $P^A = \bar{P}^A + \tilde{P}^A$ are the canonical momenta conjugate to the field variables $Q_A = \bar{Q}_A + \tilde{Q}_A$, then transition to the linearized theory implies $\{\tilde{Q}_A, \tilde{P}^B\} = \delta_A^B$. In other words, the basic phase space variables of the linearized theory are

$$\tilde{Q}_A = (\tilde{b}^i{}_\mu, \tilde{\omega}^i{}_\mu, \tilde{\lambda}^i{}_\mu, \tilde{f}^i{}_\mu), \quad \tilde{P}^A = (\tilde{\pi}_i{}^\mu, \tilde{\Pi}_i{}^\mu, \tilde{p}_i{}^\mu, \tilde{P}_i{}^\mu).$$

From the Lagrangian (2.3), we obtain the primary constraints of the linearized theory:

$$\begin{aligned}\phi_i{}^0 &:= \tilde{\pi}_i{}^0 \approx 0, & \phi_i{}^\alpha &:= \tilde{\pi}_i{}^\alpha - \varepsilon^{0\alpha\beta} \tilde{\lambda}_{i\beta} \approx 0, \\ \Phi_i{}^0 &:= \tilde{\Pi}_i{}^0 \approx 0, & \Phi_i{}^\alpha &:= \tilde{\Pi}_i{}^\alpha - 2a\varepsilon^{0\alpha\beta} \left(\sigma \tilde{b}_{i\beta} + \frac{1}{2m^2} \tilde{f}_{i\beta} \right) \approx 0, \\ \tilde{p}_i{}^\mu &\approx 0, & \tilde{P}_i{}^\mu &\approx 0.\end{aligned}\tag{3.1}$$

Total Hamiltonian. Inspired by the results of [11], we find that the quadratic canonical Hamiltonian \mathcal{H}_c can be represented in the form (up to a divergence):

$$\begin{aligned}\mathcal{H}_c = & \tilde{b}^i{}_0 \mathcal{H}_i + \tilde{\omega}^i{}_0 \mathcal{K}_i + \tilde{f}^i{}_0 \mathcal{R}_i + \tilde{\lambda}^i{}_0 \mathcal{T}_i \\ & + \bar{b}^i{}_0 \mathcal{A}_i + \bar{\omega}^i{}_0 \mathcal{B}_i + \bar{f}^i{}_0 \mathcal{C}_i + \frac{a}{m^2} (b\mathcal{V}_K)^{(2)} .\end{aligned}\quad (3.2)$$

The components of \mathcal{H}_c are defined as follows:

$$\begin{aligned}\mathcal{H}_i &:= -\varepsilon^{0\alpha\beta} \left(2a\sigma \bar{\nabla}_\alpha \tilde{\omega}_{i\beta} - 2a\Lambda_0 \varepsilon_{ijk} \bar{b}^j{}_\alpha \tilde{b}^k{}_\beta + \bar{\nabla}_\alpha \tilde{\lambda}_{i\beta} \right) , \\ \mathcal{K}_i &:= -\varepsilon^{0\alpha\beta} \left[a \bar{\nabla}_\alpha \left(2\sigma \tilde{b}_{i\beta} + \frac{1}{m^2} \tilde{f}_{i\beta} \right) + a \left(2\sigma - \frac{\Lambda_{\text{eff}}}{m^2} \right) \varepsilon_{ijk} \bar{b}^j{}_\alpha \tilde{\omega}^k{}_\beta + \varepsilon_{ijk} \bar{b}^j{}_\alpha \tilde{\lambda}^k{}_\beta \right] , \\ \mathcal{R}_i &:= -\frac{a}{m^2} \varepsilon^{0\alpha\beta} \bar{\nabla}_\alpha \tilde{\omega}_{i\beta} , \\ \mathcal{T}_i &:= -\varepsilon^{0\alpha\beta} \left(\bar{\nabla}_\alpha \tilde{b}_{i\beta} + \varepsilon_{ijk} \tilde{\omega}^j{}_\alpha \bar{b}^k{}_\beta \right) , \\ \mathcal{A}_i &:= -\varepsilon^{0\alpha\beta} \varepsilon_{ijk} \left(a\sigma \tilde{\omega}^j{}_\alpha \tilde{\omega}^k{}_\beta - a\Lambda_0 \bar{b}^j{}_\alpha \tilde{b}^k{}_\beta + \tilde{\omega}^j{}_\alpha \tilde{\lambda}^k{}_\beta \right) , \\ \mathcal{B}_i &:= -\varepsilon^{0\alpha\beta} \varepsilon_{ijk} \left(2a\sigma \bar{b}^j{}_\alpha \tilde{\omega}^k{}_\beta + \frac{a}{m^2} \tilde{\omega}^j{}_\alpha \tilde{f}^k{}_\beta + \bar{b}^j{}_\alpha \tilde{\lambda}^k{}_\beta \right) , \\ \mathcal{C}_i &:= -\frac{a}{2m^2} \varepsilon^{0\alpha\beta} \varepsilon_{ijk} \tilde{\omega}^j{}_\alpha \tilde{\omega}^k{}_\beta .\end{aligned}\quad (3.3)$$

In order to simplify further exposition, we find it more convenient to continue our analysis in a reduced phase space formalism. The formalism is based on using the 24 second class constraints $X_A = (\phi_i^\alpha, \Phi_i^\alpha, \tilde{p}_i^\alpha, \tilde{P}_i^\alpha)$ to eliminate the momenta $(\tilde{\pi}_i^\alpha, \tilde{\Pi}_i^\alpha, \tilde{p}_i^\alpha, \tilde{P}_i^\alpha)$. The dimension of the resulting reduced phase space R_1 is $N = 72 - 24 = 48$, and its structure is defined by the basic nontrivial Dirac brackets (DB):

$$\begin{aligned}\{\tilde{b}^i{}_\alpha, \tilde{\lambda}^j{}_\beta\}_1^* &= \eta^{ij} \varepsilon_{0\alpha\beta} \delta , & \{\tilde{\omega}^i{}_\alpha, \tilde{f}^j{}_\beta\}_1^* &= \frac{m^2}{a} \eta^{ij} \varepsilon_{0\alpha\beta} \delta , \\ \{\tilde{\lambda}^i{}_\alpha, \tilde{f}^j{}_\beta\}_1^* &= -2m^2 \sigma \eta^{ij} \varepsilon_{0\alpha\beta} \delta ,\end{aligned}\quad (3.4)$$

while the remaining DBs remain the same as the corresponding Poisson brackets. In R_1 , the total Hamiltonian takes the form:

$$\mathcal{H}_T = \mathcal{H}_c + u^i{}_0 \phi_i^0 + v^i{}_0 \Phi_i^0 + w^i{}_0 \tilde{p}_i^0 + z^i{}_0 \tilde{P}_i^0 .\quad (3.5)$$

Secondary constraints. The consistency conditions of the primary constraints $\tilde{\pi}_i^0, \tilde{\Pi}_i^0, \tilde{p}_i^0$ and \tilde{P}_i^0 produce the secondary constraints:

$$\begin{aligned}\hat{\mathcal{H}}_i &:= \mathcal{H}_i + \frac{a}{m^2} W_i^0 \approx 0 , \\ \mathcal{K}_i &\approx 0 ,\end{aligned}\quad (3.6a)$$

$$\begin{aligned}\hat{\mathcal{R}}_i &:= \mathcal{R}_i + \frac{a\bar{b}}{2m^2} \left[(\eta_{ij} \bar{g}^{0\mu} - \bar{h}_i^0 \bar{h}_j^\mu) (\tilde{f}^j{}_\mu + \Lambda_{\text{eff}} \tilde{b}^j{}_\mu) \right. \\ &\quad \left. + 2\Lambda_{\text{eff}} (\bar{h}_i^0 \bar{h}_j^\mu - \bar{h}_i^\mu \bar{h}_j^0) \tilde{b}^j{}_\mu \right] \approx 0 , \\ \mathcal{T}_i &\approx 0 .\end{aligned}\quad (3.6b)$$

Tertiary constraints. Let us now introduce the change of variables:

$$z^i_{0'} = z^i_0 - \bar{f}^i_m u^m_0, \quad \tilde{\pi}_i^{0'} = \tilde{\pi}_i^0 + \bar{f}_i^k \tilde{P}_k^0, \quad (3.7)$$

such that

$$u^i_0 \tilde{\pi}_i^0 + z^i_0 \tilde{P}_i^0 = u^i_0 \tilde{\pi}_i^{0'} + z^i_{0'} \tilde{P}_i^0.$$

The consistency conditions of \mathcal{K}_i and $\hat{\mathcal{R}}_i$ determine two components $z'_{\beta 0} := \bar{b}^k_{\beta} z'_{k0}$ of z'_{k0} :

$$z'_{\beta 0} = -\varepsilon_{ijk} \bar{b}^i_0 \bar{\omega}^j_0 (\tilde{f}^k_{\beta} + \Lambda_{\text{eff}} \tilde{b}^k_{\beta}) + \bar{b}_{i0} \bar{\nabla}_{\beta} (\tilde{f}^i_0 + \Lambda_{\text{eff}} \tilde{b}^i_0) + \frac{m^2}{a} \varepsilon_{ijk} \bar{b}^i_0 \bar{b}^j_{\beta} \tilde{\lambda}^k_0,$$

while the consistency of $\hat{\mathcal{H}}_i$ and \mathcal{T}_i leads to the tertiary constraints:

$$\theta_{\mu\nu} := \tilde{f}_{\mu\nu} - \tilde{f}_{\nu\mu} \approx 0, \quad (3.8a)$$

$$\psi_{\mu\nu} := \tilde{\lambda}_{\mu\nu} - \tilde{\lambda}_{\nu\mu} \approx 0, \quad (3.8b)$$

where

$$\tilde{f}_{\mu\nu} = \bar{b}^i_{\mu} \tilde{f}_{i\nu} - \Lambda_{\text{eff}} \bar{b}^i_{\nu} \tilde{b}_{i\mu}, \quad \tilde{\lambda}_{\mu\nu} = \bar{b}^i_{\mu} \tilde{\lambda}_{i\nu}. \quad (3.8c)$$

Quartic constraints. Further consistency conditions determine two components $w_{\beta 0} := \bar{b}^k_{\beta} w_{k0}$ of w_{k0} :

$$\begin{aligned} w_{\beta 0} = & -\varepsilon_{ijk} \bar{b}^i_0 \bar{\omega}^j_0 \tilde{\lambda}^k_{\beta} + \bar{b}^i_0 \bar{\nabla}_{\beta} \tilde{\lambda}^i_0 - 2\Lambda_0 \varepsilon_{ijk} \bar{b}^i_0 \bar{b}^j_{\beta} \tilde{b}^k_0 + \frac{a}{m^2} \varepsilon_{0\beta\alpha} \bar{b}^i_0 W_i^{\alpha} \\ & - a\sigma \bar{b} \varepsilon_{0\beta\alpha} \left(\bar{b}_{i0} \bar{g}^{\alpha\nu} (\tilde{f}^i_{\nu} + \Lambda_{\text{eff}} \tilde{b}^i_{\nu}) - 2\Lambda_{\text{eff}} \bar{h}_i^{\alpha} \tilde{b}^i_0 \right), \end{aligned}$$

and produce the relations

$$\chi := \bar{h}_i^{\mu} \tilde{\lambda}^i_{\mu} \approx 0, \quad (3.9a)$$

$$\varphi := \left(\sigma + \frac{\Lambda_{\text{eff}}}{2m^2} \right) \bar{h}_i^{\mu} \left(\tilde{f}^i_{\mu} + \Lambda_{\text{eff}} \tilde{b}^i_{\mu} \right) \approx 0. \quad (3.9b)$$

At the critical point (2.6), the expression φ identically vanishes, and the only quartic constraint is χ .

We close the consistency procedure by noting that the consistency of χ determines the multiplier $w_{00} := \bar{b}^k_0 w_{k0}$:

$$\bar{g}^{00} w_{00} = - \left(2\bar{g}^{\alpha 0} w_{\alpha 0} + \bar{g}^{\alpha\beta} \bar{b}^i_{\alpha} \dot{\tilde{\lambda}}_{i\beta} \right),$$

where $\dot{\tilde{\lambda}}_{i\beta}$ is calculated in Appendix C, while the absence of φ implies that $z'_{00} := \bar{b}^k_0 z'_{k0}$ remains undetermined.

- In comparison to the nonlinear BHT massive gravity, the linearized theory has *one constraint less* (φ) and *one undetermined multiplier more* (z'_{00}), which leads to a significant modification of its canonical structure.

4 Classification of constraints

Among the primary constraints, those that appear in \mathcal{H}_T with arbitrary multipliers (u^i_0, v^i_0 and z'^0_{00}) are first class (FC):

$$\tilde{\pi}_i^{0'}, \tilde{\Pi}_i^0, \tilde{P}^{00} = \text{FC}, \quad (4.1)$$

while the remaining ones, \tilde{p}_i^0 and $\tilde{P}^{\alpha 0}$, are second class. Note that $\tilde{P}^{00} := \bar{h}^{k0} \tilde{P}_k^0$.

Going to the secondary constraints, we use the following simple theorem:

- If ϕ is a FC constraint, then $\{\phi, H_T\}_1^*$ is also a FC constraint.

The proof relies on using the Jacoby identity. The theorem implies that the secondary constraints $\hat{\mathcal{H}}'_i := -\{\tilde{\pi}_i^{0'}, H_T\}_1^*$, $\mathcal{K}_i = -\{\tilde{\Pi}_i^0, H_T\}_1^*$ and $\hat{\mathcal{R}}^{00} := -\{\tilde{P}^{00}, H_T\}_1^*$ are FC. A straightforward calculation yields:

$$\begin{aligned} \hat{\mathcal{H}}'_i &= \hat{\mathcal{H}}_i + \bar{f}_i^k \hat{\mathcal{R}}_k, \\ \hat{\mathcal{R}}^{00} &= \bar{h}_i^0 \hat{\mathcal{R}}^i - \bar{h}_i^0 \bar{\nabla}_\beta (\bar{b}^i_0 \tilde{P}^{\beta 0}) + \frac{a}{2m^2} \bar{b}_{\varepsilon 0\alpha\beta} \frac{\bar{g}^{0\alpha}}{\bar{g}^{00}} (\bar{f}^{0\beta} - \bar{g}^{00} \bar{f}_0^\beta) \tilde{p}^{00} \\ &\quad - a \bar{b}_{\varepsilon 0\alpha\beta} \left[\sigma \bar{g}^{0\beta} + \frac{1}{2m^2} (\bar{f}^{0\beta} - \bar{g}^{0\beta} \bar{f} + \bar{g}^{0\beta} \bar{f}_0^0 - \bar{g}^{00} \bar{f}_0^\beta) \right] \tilde{p}^{\alpha 0}. \end{aligned}$$

Since the background is maximally symmetric, we have:

$$\begin{aligned} \hat{\mathcal{H}}'_i &= \hat{\mathcal{H}}_i - \Lambda_{\text{eff}} \hat{\mathcal{R}}_i, \\ \hat{\mathcal{R}}^{00} &= \bar{h}_i^0 \hat{\mathcal{R}}^i - \bar{h}_i^0 \bar{\nabla}_\beta (\bar{b}^i_0 \tilde{P}^{\beta 0}). \end{aligned} \quad (4.2)$$

After identifying the above 14 FC constraints, we now turn our attention to the remaining (tertiary and quartic) 17 constraints. However, we know [10] that the number of second class constraints has to be even. As one can verify, the constraint $\tilde{\psi}_{\alpha\beta}$ is FC, while the other 16 constraints are second class (Appendix D). The complete classification of constraints in the reduced space R_1 is displayed in Table 1.

Table 1. Classification of constraints in R_1		
	First class	Second class
Primary	$\tilde{\pi}_i^{0''}, \tilde{\Pi}_i^0, \tilde{P}^{00}$	$\tilde{p}_i^0, \tilde{P}^{\alpha 0}$
Secondary	$\hat{\mathcal{H}}'_i, \mathcal{K}_i, \hat{\mathcal{R}}^{00}$	$\mathcal{T}_i, \hat{\mathcal{R}}^{\alpha'}$
Tertiary	$\psi_{\alpha\beta}$	$\theta_{0\alpha}, \theta_{\alpha\beta}, \psi_{0\alpha}$
Quartic		χ

Here, $\hat{\mathcal{R}}^{\alpha'} = \bar{h}_i^\alpha \hat{\mathcal{R}}^{i'}$, where $\hat{\mathcal{R}}^{i'}$ is a suitable modification of $\hat{\mathcal{R}}_i$, defined so that it does not contain \tilde{f}_{i0} :

$$\begin{aligned} \hat{\mathcal{R}}^{i'} &:= \hat{\mathcal{R}}_i - \frac{a\bar{b}}{4m^2} (\bar{h}_i^\mu \bar{g}^{0\nu} - \bar{h}_i^\nu \bar{g}^{0\mu}) \theta_{\mu\nu} \\ &\equiv \mathcal{R}_i + \frac{a\bar{b}}{2m^2} (\bar{h}_i^\alpha \bar{h}_j^0 - \bar{h}_i^0 \bar{h}_j^\alpha) (\tilde{f}^j_\alpha - \Lambda_{\text{eff}} \tilde{b}^j_\alpha). \end{aligned} \quad (4.3)$$

Now, we can calculate the number of independent dynamical degrees of freedom with the help of the standard formula:

$$N^* = N - 2N_1 - N_2 ,$$

where N is the number of phase space variables in R_1 , N_1 is the number of FC, and N_2 the number of second class constraints. Using $N = 48$ and, according to the results in Table 1, $N_1 = 15$ and $N_2 = 16$, we obtain that

- the number of physical modes in the phase space is $N^* = 2$, and consequently, the BHT theory at the critical point (2.6) exhibits one Lagrangian degree of freedom.

5 Extra gauge symmetry

The presence of an extra primary FC constraint \tilde{P}^{00} implies the existence of an extra gauge symmetry. To simplify its canonical construction, we go over to the reduced phase space R_2 , which is obtained from R_1 by using the additional constraints

$$R_2 : \quad \theta_{\beta 0} \equiv \tilde{f}_{\beta 0} - \tilde{f}_{0\beta} = 0 , \quad \tilde{P}^{\beta 0} = 0 , \quad (5.1)$$

to eliminate the variables $\tilde{f}_{\beta 0}$ and $\tilde{P}^{\beta 0}$. Basic DBs between the canonical variables in R_2 retain the same form as in R_1 . Starting with the primary FC constraint \tilde{P}^{00} , Castellani's algorithm [19] leads to the following canonical generator in R_2 :

$$\begin{aligned} G_E = & -2\tilde{\epsilon}\tilde{P}^{00} + \dot{\epsilon} \left[-2\hat{\mathcal{R}}^{0'} + 2(\bar{h}^{i0}\bar{\nabla}_0\bar{b}^i{}_0)\tilde{P}^{00} + \varepsilon_{ijk}\bar{h}^{i0}\bar{b}^j{}_0\tilde{\Pi}^k{}_0 \right] \\ & + \epsilon \left[\varepsilon^{0\alpha\beta}\bar{b}^i{}_\alpha\tilde{\lambda}_{i\beta} + \tilde{\pi}_0{}^0 - \varepsilon_{ijk}\bar{\nabla}_\alpha(\bar{h}^{i\alpha}\bar{b}^j{}_0\tilde{\Pi}^k{}_0) + 2\bar{\nabla}_\alpha\hat{\mathcal{R}}^{\alpha'} \right. \\ & \left. - 2\bar{\nabla}_\alpha(\bar{h}_i{}^\alpha\tilde{P}^{00}\bar{\nabla}_0\bar{b}^i{}_0) + \Lambda_{\text{eff}}\bar{g}_{00}\tilde{P}^{00} \right] . \end{aligned} \quad (5.2)$$

The action of G_E on R_2 is given by $\delta_E\phi = \{\phi, G_E\}_2^*$, which yields:

$$\begin{aligned} \delta_E\tilde{b}^i{}_\mu &= \epsilon\bar{b}^i{}_\mu , \\ \delta_E\tilde{\omega}^i{}_\mu &= -\varepsilon^{ijk}\bar{b}_{j\mu}\bar{h}_k{}^\nu\bar{\nabla}_\nu\epsilon , \\ \delta_E\tilde{f}^i{}_\alpha &= -2\bar{\nabla}_\alpha(\bar{h}^{i\nu}\bar{\nabla}_\nu\epsilon) + \Lambda_{\text{eff}}\epsilon\bar{b}^i{}_\alpha , \\ \delta_E\tilde{f}_{00} &= -2b_{i0}\bar{\nabla}_0(\bar{h}^{i\nu}\bar{\nabla}_\nu\epsilon) , \\ \delta_E\tilde{\lambda}^i{}_\mu &= 0 . \end{aligned} \quad (5.3a)$$

To make a comparison with (2.5), we now derive the transformation law for the variable $\tilde{f}^i{}_0 = \bar{h}_i{}^\mu\tilde{f}_{\mu 0} + \Lambda_{\text{eff}}\bar{h}_i{}^\nu\bar{b}^j{}_0\bar{b}_{j\nu}$. Using

$$\begin{aligned} \delta_E\tilde{f}_{\beta 0} &= -2\bar{b}^i{}_0\bar{\nabla}_\beta(\bar{h}_i{}^\nu\bar{\nabla}_\nu\epsilon) \\ &= -2\partial_\beta\partial_0\epsilon - 2\bar{b}^i{}_0(\bar{\nabla}_\beta\bar{h}_i{}^\nu)\partial_\nu\epsilon \\ &= -2\partial_0\partial_\beta\epsilon - 2\bar{b}^i{}_\beta(\bar{\nabla}_0\bar{h}_i{}^\nu)\partial_\nu\epsilon \\ &= -2\bar{b}^i{}_\beta\bar{\nabla}_0(\bar{h}_i{}^\nu\bar{\nabla}_\nu\epsilon) , \end{aligned}$$

one obtains

$$\delta_E\tilde{f}^i{}_0 = -2\bar{\nabla}_0(\bar{h}^{i\nu}\bar{\nabla}_\nu\epsilon) + \Lambda_{\text{eff}}\epsilon\bar{b}^i{}_0 . \quad (5.3b)$$

The transformation rules (5.3) are in complete agreement with (2.5).

6 Concluding remarks

In the *nonperturbative* regime of the BHT gravity, the constraint structure is found to depend critically on the value of Ω^{00} , where $\Omega^{\mu\nu} = \sigma g^{\mu\nu} + G^{\mu\nu}/2m^2$ [11]. In the region of the phase space where $\Omega^{00} \neq 0$, the BHT theory has *two* Lagrangian degrees of freedom, which corresponds to two helicity states of the massive graviton excitation.

In this paper, we studied the canonical structure of the BHT gravity *linearized* around the maximally symmetric background, $G^{\mu\nu} = \Lambda_{\text{eff}} g^{\mu\nu}$. At the critical point $\Lambda_0/m^2 = -1$, the background solution is characterized by the property $\Omega^{\mu\nu} = 0$, the covariant version of $\Omega^{00} = 0$. Analyzing the constraint structure of the linearized theory, we constructed the canonical generator of *extra gauge symmetry*, which is responsible for transforming two massive graviton excitations into a single, partially massless mode; moreover, the theory is found to have *one* Lagrangian degree of freedom.

In order to properly understand the linearized theory, one should stress that although we have $\Omega^{\mu\nu} = 0$ on the very background, the linearized theory is well-defined in the region off the background, where $\Omega^{\mu\nu} \neq 0$. In this region, the process of linearization induces a drastic modification of the canonical structure of the BHT theory, leading to the change of the number and type of constraints and physical degrees of freedom.

Thus, the canonical structure of the BHT gravity at the critical point $\Lambda_0/m^2 = -1$ does not remain the same after linearization. Following the arguments of Chen et al. [20], we are led to conclude that the canonical consistency of the BHT gravity, expressed by the stability of its canonical structure under linearization, is violated at the critical point $\Lambda_0/m^2 = -1$.

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A Maximally symmetric solutions

Variation of the action (2.1) with respect to the basic dynamical variables $b^i_\mu, \omega^i_\mu, f^i_\mu$ and λ^i_μ , yields the following field equations [11]:

$$\begin{aligned} a\varepsilon^{\mu\nu\rho} (\sigma R_{i\nu\rho} - \Lambda_0 \varepsilon_{ijk} b^j_\nu b^k_\rho) - \frac{ab}{m^2} \mathcal{T}_i^\mu + \varepsilon^{\mu\nu\rho} \nabla_\nu \lambda_{i\rho} &= 0, \\ \varepsilon^{\mu\nu\rho} \left(a T_{i\nu\rho} + \frac{a}{m^2} \nabla_\nu f_{i\rho} + \varepsilon_{ijk} b^j_\nu \lambda^k_\rho \right) &= 0, \\ \varepsilon^{\mu\nu\rho} R_{i\nu\rho} - b (f_i^\mu - f h_i^\mu) &= 0, \\ \varepsilon^{\mu\nu\rho} T_{i\nu\rho} &= 0, \end{aligned} \tag{A.1}$$

where \mathcal{T}_i^μ is the energy-momentum tensor associated to \mathcal{L}_K ,

$$\mathcal{T}_i^\mu := -\frac{1}{b} \frac{\partial \mathcal{L}_K}{\partial b^i_\mu} = h_i^\mu \mathcal{V}_K - \frac{1}{2} (f_{ik} f^{k\mu} - f f_i^\mu).$$

The last equation ensures that spacetime is Riemannian, the third and the second one imply

$$f_{ij} = 2L_{ij}, \quad \lambda_{ij} = \frac{2a}{m^2} C_{ij},$$

where L_{ij} and C_{ij} are the Schouten and the Cotton tensor, respectively:

$$L_{ij} = \hat{R}_{ij} - \frac{1}{4}\eta_{ij}R, \quad C_{ij} = \varepsilon_i^{mn}\nabla_m L_{nj},$$

and the first field equation takes the form:

$$\sigma G_{ij} - \Lambda_0 \eta_{ij} - \frac{1}{2m^2} K_{ij} = 0, \quad (\text{A.2})$$

where $K_{ij} := \mathcal{T}_{ij} - 2(\nabla_m C_{in})\varepsilon^{mn}_j$. We display here also a set of algebraic consequences of the field equations:

$$\begin{aligned} f_{\mu\nu} &= f_{\nu\mu}, \\ \lambda_{\mu\nu} &= \lambda_{\nu\mu}, \quad \lambda = 0, \\ \sigma f + 3\Lambda_0 + \frac{1}{2m^2} \mathcal{V}_K &= 0. \end{aligned} \quad (\text{A.3})$$

For maximally symmetric solutions with $\bar{G}_{ij} = \Lambda_{\text{eff}} \eta_{ij}$, we have $\bar{L}_{ij} = -\frac{1}{2}\Lambda_{\text{eff}} \eta_{ij}$, $\bar{C}_{ij} = 0$, and consequently:

$$\bar{f}_{ij} = -\Lambda_{\text{eff}} \eta_{ij}, \quad \bar{\lambda}_{ij} = 0. \quad (\text{A.4})$$

Then, the last equation in (A.3) gives the following solution for Λ_{eff} :

$$\Lambda_{\text{eff}} = -2m^2 \left(\sigma \pm \sqrt{1 + \Lambda_0/m^2} \right). \quad (\text{A.5})$$

For $\Lambda_0/m^2 = -1$, the ground state is uniquely defined by $\Lambda_{\text{eff}} = -2m^2\sigma$.

B Lagrangian form of extra gauge symmetry

In this appendix, we show that the linearized field equations (2.4) are invariant under the extra gauge symmetry (2.7). The last two equations in (2.4) are invariant for all values of the parameters. Denoting the left-hand sides of the first two field equations by F_1 and F_2 , respectively, we find:

$$\begin{aligned} \delta_E F_1 &= -4a \left(\Lambda_0 + \frac{\Lambda_{\text{eff}}^2}{4m^2} \right) \bar{b} \bar{h}_i{}^\mu \epsilon + a \left(2\sigma + \frac{\Lambda_{\text{eff}}}{m^2} \right) \varepsilon^{\mu\nu\rho} \varepsilon_{ijk} \bar{b}^j{}_\nu \bar{\nabla}_\rho (\bar{h}^{k\sigma} \bar{\nabla}_\sigma \epsilon), \\ \delta_E F_2 &= -2a \left(2\sigma + \frac{\Lambda_{\text{eff}}}{m^2} \right) \varepsilon^{\mu\nu\rho} \bar{b}_{i\nu} \bar{\nabla}_\rho \epsilon. \end{aligned}$$

The corresponding conditions of invariance,

$$2\sigma + \frac{\Lambda_{\text{eff}}}{m^2} = 0, \quad \Lambda_0 + \frac{\Lambda_{\text{eff}}^2}{4m^2} = 0,$$

are both equivalent to the critical condition (2.6).

C Calculation of the determined multipliers

In the process of calculating the values of the multipliers $z'_{\beta 0}$, $w_{\beta 0}$ and w_{00} , we need the DBs of $\tilde{Q}_A = (\tilde{b}^i_\alpha, \tilde{\omega}^i_\alpha, \tilde{\lambda}^i_\alpha, \tilde{f}^i_\alpha)$ with the total Hamiltonian:

$$\begin{aligned}
\dot{\tilde{b}}^i_\alpha &= -\varepsilon^{ijk}(\bar{\omega}_{j0}\tilde{b}_{k\alpha} + \tilde{\omega}_{j0}\bar{b}_{k\alpha} - \tilde{\omega}_{j\alpha}\bar{b}_{k0}) + \bar{\nabla}_\alpha \tilde{b}^i_0, \\
\dot{\tilde{\omega}}^i_\alpha &= -\varepsilon^{ijk}\bar{\omega}_{j0}\tilde{\omega}_{k\alpha} + \bar{\nabla}_\alpha \tilde{\omega}^i_0 \\
&\quad + \frac{1}{2}\bar{b}\varepsilon_{0\alpha\beta} \left[(\delta^i_j \bar{g}^{\beta\nu} - \bar{h}^{i\beta} \bar{h}_j{}^\nu)(\tilde{f}^j_\nu + \Lambda_{\text{eff}} \tilde{b}^j_\nu) + 2\Lambda_{\text{eff}} (\bar{h}^{i\beta} \bar{h}_j{}^\nu - \bar{h}^{i\nu} \bar{h}_j{}^\beta) \tilde{b}^j_\nu \right], \\
\dot{\tilde{\lambda}}^i_\alpha &= -\varepsilon^{ijk}\bar{\omega}_{j0}\tilde{\lambda}_{k\alpha} + \bar{\nabla}_\alpha \tilde{\lambda}^i_0 + 2\Lambda_0 \varepsilon^{ijk}(\bar{b}_{j0}\tilde{b}_{k\alpha} - \bar{b}_{j\alpha}\tilde{b}_{k0}) + \frac{a}{m^2} \varepsilon_{0\alpha\beta} \tilde{W}_i{}^\beta \\
&\quad - a\sigma \bar{b}\varepsilon_{0\alpha\beta} \left[(\delta^i_j \bar{g}^{\beta\nu} - \bar{h}^{i\beta} \bar{h}_j{}^\nu)(\tilde{f}^j_\nu + \Lambda_{\text{eff}} \tilde{b}^j_\nu) + 2\Lambda_{\text{eff}} (\bar{h}^{i\beta} \bar{h}_j{}^\nu - \bar{h}^{i\nu} \bar{h}_j{}^\beta) \tilde{b}^j_\nu \right], \\
\dot{\tilde{f}}^i_\alpha &= -\varepsilon^{ijk}(\bar{\omega}_{j0}\tilde{f}_{k\alpha} + \tilde{\omega}_{j0}\bar{f}_{k\alpha} - \tilde{\omega}_{j\alpha}\bar{f}_{k0}) + \bar{\nabla}_\alpha \tilde{f}^i_0 - \frac{m^2}{a} \varepsilon^{ijk}(\bar{b}_{j0}\tilde{\lambda}_{k\alpha} - \bar{b}_{j\alpha}\tilde{\lambda}_{k0}).
\end{aligned}$$

D Second class constraints

In this appendix, we show that the set of 16 constraints in the second column of Table 1 is of the second class. Instead of calculating the determinant of the 16×16 matrix of the related DBs, the proof is derived iteratively.

Step 1. We begin with the subset of constraints $Y_A := (\psi_{\beta 0}, \chi, \tilde{p}^{\alpha 0}, \tilde{p}_0^0)$. The corresponding 6×6 matrix Δ_1 with matrix elements $\{Y_A, Y_B\}_1^*$ reads:

$$\Delta_1 = \begin{pmatrix} 0_{3 \times 3} & -I_{3 \times 3} \\ -I_{3 \times 3}^T & 0_{3 \times 3} \end{pmatrix},$$

where I is the unit matrix. The matrix Δ_1 is regular, $\det \Delta_1 = 1$.

Step 2. Next, we consider the subset of constraints $Z_A := (\theta_{0\beta}, P^{\alpha 0})$. The corresponding 4×4 matrix

$$\Delta_2 = \begin{pmatrix} 0_{2 \times 2} & \delta_\beta^\alpha \\ -\delta_\beta^\alpha & 0_{2 \times 2} \end{pmatrix}$$

is regular, since $\det \Delta_2 = 1$.

Step 3. Finally, we consider the remaining subset $W_A = (\mathcal{T}_i, \hat{\mathcal{R}}^{\alpha'}, \frac{1}{2}\varepsilon^{0\alpha\beta}\theta_{\alpha\beta})$. The 6×6 matrix $\{W_A, W_B\}_1^*$ takes the form

$$\Delta_3 = \begin{pmatrix} 0_{3 \times 3} & D_{3 \times 2} & E_{3 \times 1} \\ -D_{2 \times 3}^T & F_{2 \times 2} & 0_{2 \times 1} \\ -E_{1 \times 3}^T & 0_{1 \times 2} & 0_{1 \times 1} \end{pmatrix},$$

where the matrices D, E and F are given by

$$\begin{aligned}
D_i{}^\alpha &:= \{\mathcal{T}_i, \hat{\mathcal{R}}^{\alpha'}\}_1^* = -\bar{b}\bar{h}^{j\alpha} \left[\varepsilon_{ijn} \left(\frac{1}{2}\bar{h}^{n0} - \bar{g}^{00}\bar{b}^n_0 \right) - \bar{h}_j{}^0 \varepsilon_{imn} \bar{b}^m_0 \bar{h}^{n0} \right] \delta, \\
E_i &:= \{\mathcal{T}_i, \frac{1}{2}\varepsilon^{0\alpha\beta}\theta_{\alpha\beta}\}_1^* = -\frac{2m^2}{a} \bar{b}\bar{h}_i{}^0 \delta, \\
F^{\alpha\beta} &:= \{\hat{\mathcal{R}}^{\alpha'}, \hat{\mathcal{R}}^{\beta'}\}_1^*.
\end{aligned}$$

The regularity of Δ_3 follows from

$$\det \Delta_3 = \left(\frac{1}{2} \varepsilon^{ijk} \varepsilon_{0\alpha\beta} E_i D_j^\alpha D_k^\beta \right)^2 = \left(\frac{m^2}{2a} \bar{b}^2 \right)^2 \neq 0.$$

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